On the decomposition of hereditary graph classes

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Hereditary class of graphs

Hereditary property: a graph property which holds for a graph and is inherited by all its *induced* subgraphs.

Induced subgraph: a subgraph H obtained from a graph G by *deleting* some vertices of G. (*We say that G contains H*)

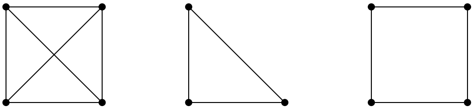

Figure: A graph, an induced subgraph, and a *non*-induced subgraph

Definition

A class of graphs is *hereditary* if it is **closed** under taking induced subgraphs.

Hereditary class of graphs

Some examples

- planar graphs;
- bipartite graphs;
- graphs of bounded degree;
- forests;
- chordal graphs;
- perfect graphs;
- line graphs;
- graphs that contain no clique of size 3;
- graphs that contain no even *hole* (i.e. chordless cycle);

Hereditary class of graphs

Any hereditary class can be characterized as the class of graphs that do not contain any graph in $\mathcal{F}$ for some family $\mathcal{F}$.

- forests = $(C_3, C_4, C_5, C_6, C_7, \ldots)$-free
- bipartite graphs = $(C_3, C_5, C_7, \ldots)$-free
- chordal graphs = $(C_4, C_5, C_6, C_7, \ldots)$-free
- perfect graphs = $(C_3, \overline{C_3}, C_5, \overline{C_5}, C_7, \overline{C_7}, \ldots)$-free
- P_4-free graphs, (P_4, C_4)-free graphs, etc...

G is ***F*-free** if no induced subgraph of G is isomorphic to F; and is **$\mathcal{F}$-free** if no induced subgraph of G is isomorphic to each $F \in \mathcal{F}$.

Remark: $\mathcal{F}$ can be a finite/infinite family.

Why forbidding induced subgraphs?

(Possibly) naive answers...

- When I started my PhD, my supervisor told me to do so;
- There are many possibilities to forbid induced subgraphs, so this could lead to publishing many papers.

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Why forbidding induced subgraphs?

In real world, many problems can be formulated as graphs (which are hereditary):

- vertices represent objects;
- edges represent constraints

Main concerns:

- How classes of graphs closed under taking induced subgraphs can be described in the most general possible way?
- What properties can be proved about them?

Many NP-Hard problems (*e.g. coloring, maximum independent set*) become easy when some configurations are forbidden. (*e.g. forests, chordal graphs, perfect graphs*).

Graph decomposition

Definition (Decomposition theorem in general...)

A decomposition theorem for a class $\mathcal{C}$ says that every object of $\mathcal{C}$ either belongs to some well-understood *basic* class, or it can be broken into smaller pieces according to some well-described rules.

Example in mathematics: "The fundamental theorem of arithmetic"

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Example in mathematics: "The fundamental theorem of arithmetic"

For a class of graphs $\mathcal{C}$, we define a set of basic graphs $\mathcal{C}_0$ and a list of graph *decomposition* operations $\mathcal{L}$, s.t. if $G \in \mathcal{C}$:

- either $G \in \mathcal{C}_0$; or
- G can be broken down to smaller graphs G' and G'' using an operation in $\mathcal{L}$

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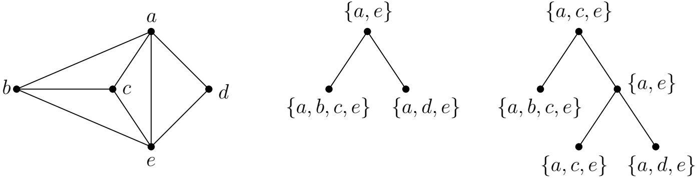
- either $G \in \mathcal{C}_0$; or
- G can be broken down to smaller graphs G' and G'' using an operation in $\mathcal{L}$
- If furthermore, every G can be built from smaller graphs G' and G'' belonging to $\mathcal{C}$ using a *compositions* operation $\mathcal{L}'$ (the "reverse" operations of $\mathcal{L}$, then it is a structure theorem).

Example of decomposition theorem

$S \subsetneq V(G)$ is a cutset of a connected graph G if $G \setminus S$ induced a disconnected graph.

Clique cutset decomposition (Tarjan, 1985)



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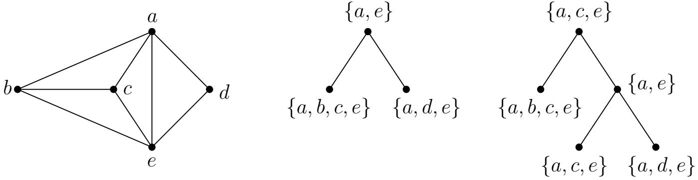

Decomposition of chordal graphs (i.e. C_k-free, for all $k \geq 4$)

Theorem (Dirac, 1961)

If G is a chordal graph, then either G is a complete graph, or G admits a clique-cutset.

Graph decomposition for algorithm (1)

Graph recognition algorithm

Given a class of graphs $\mathcal{C}$. How do we decide if a given input graph G is in $\mathcal{C}$?

1. Get a decomposition theorem of $\mathcal{C}$;
2. decompose G until no decomposition is possible;
3. check if all graphs obtained from the decomposition are basic graphs of $\mathcal{C}$.

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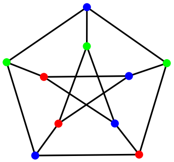
How do we ensure that the algorithm is in poly-time?

- decomposition procedure takes poly-time
- the prescribed *compositions* are class-preserving
- the number of graphs to check is polynomial

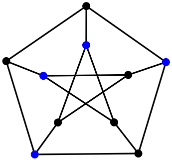
Graph decomposition for algorithm (2)

Applied to combinatorial problems

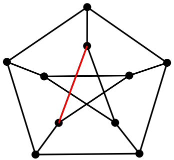
- **Vertex coloring:** assignment of (as minimum possible) colors to the vertices, no adjacent vertices receive the same color
- **Maximum independent set:** finding set of pairwise non-adjacent vertices with maximum cardinality
- **Maximum clique:** finding set of pairwise adjacent vertices with maximum cardinality



coloring
chromatic number : χ



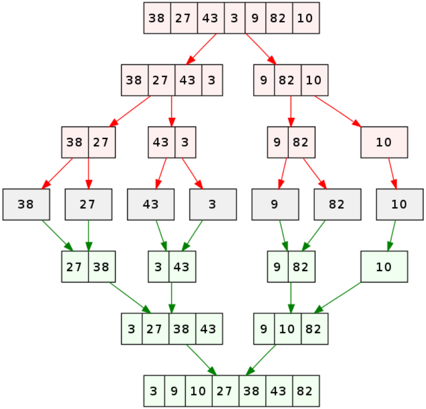
max independent set
independent set number: α



maximum clique
clique number : ω

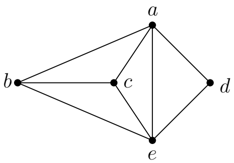
Graph decomposition for algorithm (3)

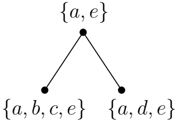
The graph-decomposition based algorithm is usually done through the divide-and-conquer approach.

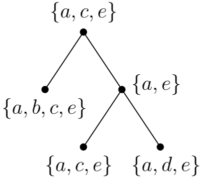


Graph decomposition for algorithm (4)

Clique cutset decomposition (Tarjan, 1985)







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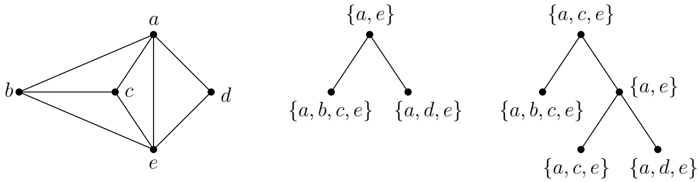
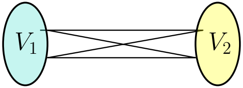


- Graph recognition is in poly-time.
- *Maximum clique*, *coloring*, and *max independent set* problems can be solved efficiently using the divide-and-conquer approach.

We can also decompose through edge-cutset

1-join



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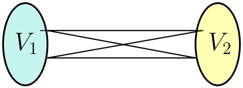
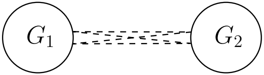
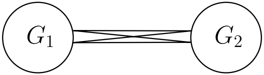


Example:

Theorem (Corneil, et al, 1981, and many others...)

If G is P_4-free (i.e. cograph), then either G is K_1, or G can be obtained from two graphs G_1 and G_2 by either a disjoint union, or a 1-join.





Maximum clique, *coloring*, and *max independent set* are poly-time.

Decomposition technique applied on perfect graphs (1)

Perfect graphs and Berge graphs

G is perfect if for every induced subgraph H of G, $\chi(H) = \omega(H)$
Berge graphs: the class of $(C_k, \overline{C_k})$-free with k is an odd number

Conjecture (Strong Perfect Graph Conjecture, Berge, 1961)

The class of perfect graphs and the class of Berge graphs are equivalent.

Decomposition technique applied on perfect graphs (2)

Decomposition theorem of Berge graphs

Theorem (Chudnovsky, Robertson, Seymour, and Thomas 2002)

Every Berge graph is basic, or has a 2-join, a complement 2-join, a homogeneous pair or a balanced skew partition.

- Basic graphs: bipartite, complement of bipartite, line graph of bipartite, complement of line graph of bipartite, and doubled graphs.

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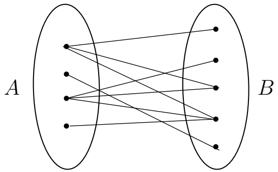

Figure: Bipartite graph

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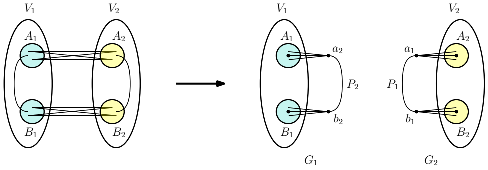

Figure: 2-join

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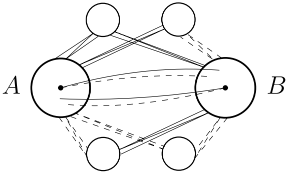

Figure: Homogeneous pair (A, B)

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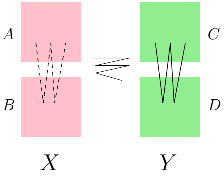

Figure: Skew partition (X, Y)

Decomposition technique applied on perfect graphs (3)

Theorem (Strong Perfect Graph Theorem, proved by Chudnovsky, Robertson, Seymour, and Thomas, 2012)

A graph is perfect if and only if it is a Berge graph.

Proof steps:

1. Theorem 1: Every Berge graphs is either basic or admits a decomposition.
2. Theorem 2: Every basic graph is perfect.
3. Theorem 3: If G is minimally imperfect, then G does not admit any of the decomposition of Thm 1.

Coloring and max ind set are in poly-time for perfect graphs, but no combinatorial use of the decomposition theorem.

Even-hole-free graphs

Decomposition theorem

Theorem (Conforti, Cornuéjols, Kapoor, and Vušković, 2002, then improved by da Silva and Vušković, 2013)

A connected even-hole-free graph (actually proved for a superclass) is either basic or it has a 2-join or a star cutset.

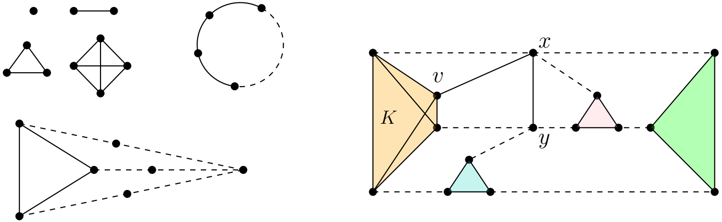

Figure: Basic graphs

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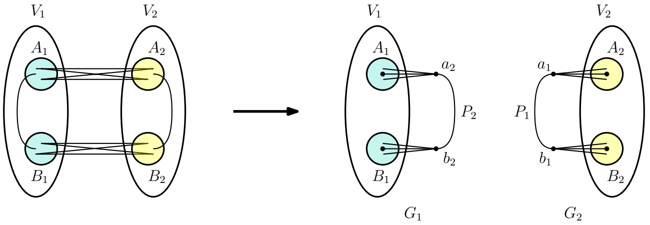

Figure: 2-join

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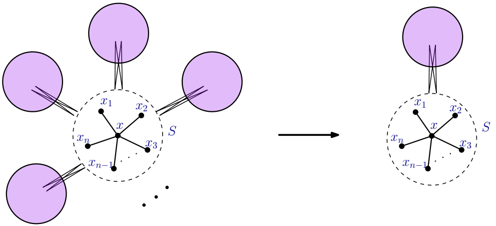


Figure: Star cutset

Even-hole-free graphs

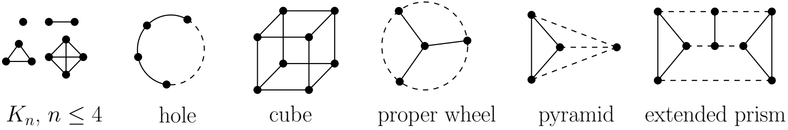
Nevertheless, the combinatorial problems are still **open** for this class!

- The decomposition theorem is applied for the recognition algorithm, but cannot be applied for solving the coloring / max ind set problems.

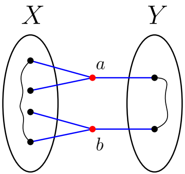
Even-hole-free graphs with $\Delta \leq 3$

$\Delta(G)$: maximum degree among $V(G)$

The basic graphs:



Cutsets: clique cutset and proper separator



Even-hole-free graphs with $\Delta \leq 3$

Theorem (Aboulker, Adler, Kim, Sintiari, Trotignon (2020))

Let G be a subcubic even-hole-free graph. Then one of the following holds:

- *G is a basic graph;*
- *G has a clique separator of size at most 2;*
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Sketch of proof.

- in each step, we "eliminate" a basic graph H
- assume that $G \in \mathcal{C}$ contains H, prove that either $G = H$, or G admits a "good separator", or G contains an obstruction
- repeat until all basic graphs are considered

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Implication: subcubic even-hole-free graphs have constant "treewidth" leading to poly-time complexity for the aforementioned combinatorial problems (coloring, max ind set).

Decomposing subcubic even-hole-free graphs (*an example*)

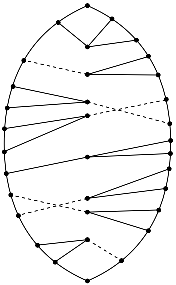

Figure: Decomposition of a non-basic subcubic even-hole-free graphs

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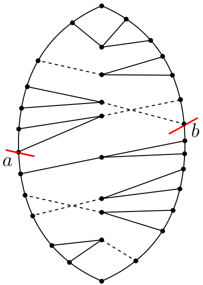

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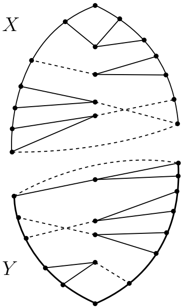

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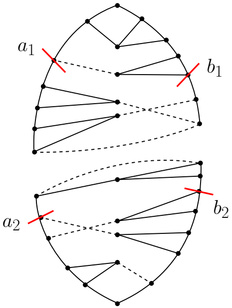

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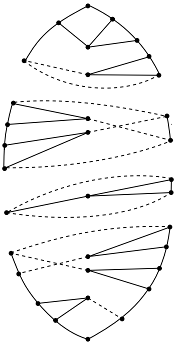

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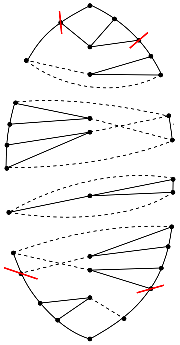

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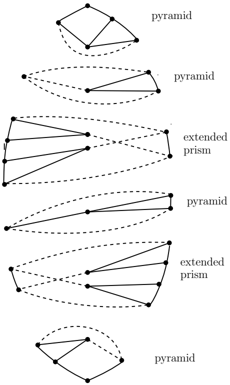

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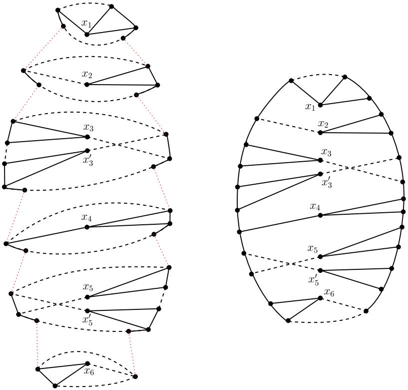
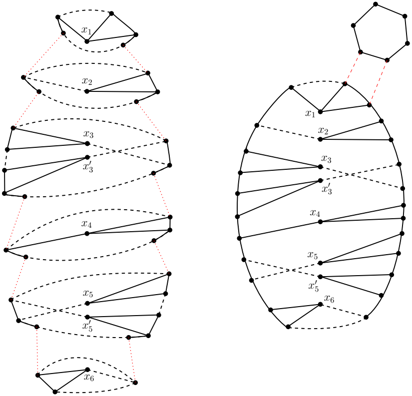


Figure: Proper gluing operations to construct a (non-basic) subcubic EHF graphs

Decomposing subcubic even-hole-free graphs (*an example*)



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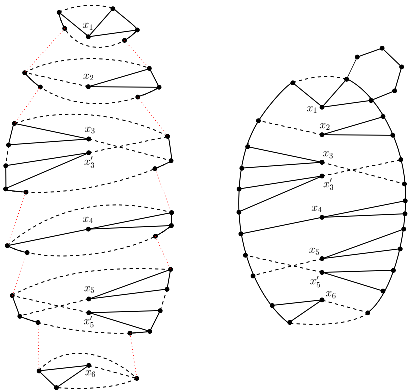

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Generalization

Conjecture

Let $\mathcal{C}$ be the class of even-hole-free graphs of maximum degree Δ. Then a decomposition theorem with a similar fashion exists for the class.

Implication: if this is true, this would lead to "treewidth bounded on Δ", which means that many combinatorial problems are in poly-time.

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Theorem (Abrishami, Chudnovsky, and Vušković, 2020)

For every $\Delta \geq 0$, there exists an integer k such that even-hole-free graphs with maximum degree at most Δ have treewidth at most k.

Implication: coloring and max ind set are polynomial (in k) for this class.

(Even hole, pyramid)-free graphs $\Delta = 4$

Theorem (Sintiari, Trotignon (2020))

Let G be an (even hole, pyramid)-free graph with $\Delta(G) \leq 4$. Then one of the following holds:

- *G is a basic graph;*
- *G has a clique separator of size at most 3;*
- *G has a proper separator for $\mathcal{C}$.*

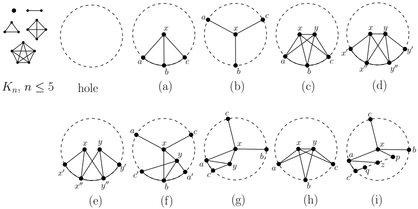

Figure: Basic graphs in the decomposition of the class

Other works

We have seen:

- Chordal graphs: forbidding all holes
- Perfect graphs: forbidding odd holes and their complement
- Even-hole-free graphs: forbidding all even holes

What if we forbid all holes that are not not of length k (for some fixed k)?

$\mathcal{C}_k$ = the class of graphs whose holes all of length k [with J. Horsfield, M. Preissmann, C. Robin, N. Trotignon, K. Vušković, and independently studied by L. Cook and P. Seymour]

What next?

1. Studying other hereditary graph classes (related to perfect graphs and even-hole-free graphs / or not related). There are still many things to explore!
2. Could we apply the decomposition theorem technique to solve graph labeling problems? Can we work on labeling on some hereditary graph classes?
3. Same question as (2) for metric dimension, Ramsey graphs, etc...

References

1. Perfect graphs: a survey (Nicolas Trotignon, 2015)
2. Even-hole-free graphs: a survey (Kristina Vušković, 2010)